# Coupling of sound and internal waves in shear flows

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(Received 27 November 1995; revised manuscript received 29 July 1996)

Gravity waves in the parallel shear flow of a continuously stratified compressible fluid are considered. It is demonstrated that the shear induces a coupling between the sound waves and the internal gravity waves. The conditions for the effectiveness of the coupling are defined. It is also shown that, under suitable conditions, beat waves can be generated. [S1063-651X(96)12012-2]

PACS number(s): 47.35.+i, 03.40.Kf, 43.20.+g, 92.60.Dj

It is well known that in a medium with a gravity-induced stratification the *buoyancy forces* tend to excite *internal gravity waves* originating from a balance between the fluid inertia and the gravitational restoring force [1,2]. The internal gravity waves (hereafter referred to as IGW's), propagating in a differentially moving fluid—that is, in a *shear flow* with continuous, gravity-induced stratification, display a rich and complex structure.

In order to study this problem it is very convenient to employ the scheme where a moving coordinate system is used and the temporal problem is examined directly. The method can, in principle, be used for any velocity profiles but it is mostly useful for ones that are piecewise linear [3–5]. Going to the moving frame mitigates the need for a Laplace transform [5–7] and greatly simplifies the solution of the initial value problem.

The problem of the evolution of IGW in an *incompressible* parallel shear flow with linear velocity profile was recently considered by Chagelishvili [8]. In that study, nonmodal algebraically growing solutions, indicating the possibility of anomalous amplification of IGW in shear flows, were readily found. This paper deals with the same problem for a compressible, unbounded, parallel flow with a uniform (linear) shear.

In [9], where the evolution of two-dimensional (2D) perturbations in a compressible, plane Couette flow was considered the mechanism of the energy exchange between the mean flow and sound-type perturbations was discovered. A *linear* mechanism of mutual transformation of waves, and a corresponding energy transfer induced by the existence of the velocity shear was found in [10] for the 2D waves in an unbounded, parallel *hydromagnetic* flow (see also [11]). It seems likely that analogous mechanisms will be operative in other kinds of parallel shear flows, where conditions for the excitation of several (more than one) wave modes exist.

Since we are dealing with the shear flow in which sound waves (SW) and IGW may be simultaneously excited, it is reasonable to expect that these modes may become effectively coupled implying a linear mutual transformation with corresponding energy transfer between the modes.

Let us consider the evolution of two-dimensional perturbations in a compressible, unbounded shear flow with a steady unidirectional mean velocity (parallel flow) that varies linearly with height. Let us choose the coordinate axes such that the regular velocity vector $\mathbf{U}_0 \equiv (Ay,0)$, is along $x$, and the acceleration due to gravity $\mathbf{g}=(0,-g_0)$ is along negative $y$. The basic system of linearized equations, describing the evolution of the small-scale, 2D perturbations in this flow, takes the form

$$D_t\rho' + \rho_0(\partial_x u_x + \partial_y u_y) + (\partial_y \rho_0)u_y = 0, \tag{1}$$

$$D_t S' + (\partial_y S_0)u_y = 0, \tag{2}$$

$$D_t u_x + A u_y = -\frac{1}{\rho_0}\partial_x P', \tag{3}$$

$$D_t u_y = -\frac{1}{\rho_0}\partial_y P' + \frac{\rho'}{\rho_0^2}\partial_y P_0, \tag{4}$$

$$\rho' = \left(\frac{\partial \rho_0}{\partial S_0}\right)_{P_0} S' + \frac{P'}{c_s^2}, \tag{5}$$

where $c_s \equiv [(\partial P_0/\partial \rho_0)]^{1/2}$, and $D_t \equiv \partial_t + Ay\partial_x$. Making use of the equilibrium condition $\partial_y P_0 = -\rho_0 g_0$, it is straightforward to eliminate $\rho'$ from (1) and (4) to yield

$$D_t P' + \rho_0 c_s^2(\partial_x u_x + \partial_y u_y) - \rho_0 g_0 u_y = 0, \tag{6}$$

$$D_t u_y = -\frac{1}{\rho_0}\partial_y P' - \frac{g_0}{\rho_0}\left(\frac{\partial \rho_0}{\partial S_0}\right)_{P_0} S' - \frac{g_0}{\rho_0 c_s^2}P'. \tag{7}$$

To "set up" the analysis, we affect the transformation, $x_1 = x - Ayt$; $y_1 = y$; $t_1 = t$, $(D_t \rightarrow \partial_{t_1}$; $\partial_y \rightarrow \partial_{y_1} - At_1\partial_{x_1})$, which effectively takes us from the laboratory to the local rest frame of the basic flow [5,9–12]. In new coordinates, where the initial inhomogeneity in space $(y)$ has been exchanged for a new inhomogeneity in time, we may expand the perturbations as

 

$F=\int dk_{x_1}dk_{y_1}\hat{F}(k_{x_1},k_{y_1},t_1)\exp[i(k_{x_1}x_1+k_{y_1}y_1)]$, and convert Eqs. (6), (2), (3), and (7) to a set of first order, ordinary differential equations for $\hat{F}(k_{x_1},k_{y_1},t_1)$, which will be hereafter referred to as spatial Fourier harmonics (SFH) [9–12]. It is convenient to write these equations in dimensionless notation: $R\equiv A/c_s k_{x_1}$, $T\equiv c_s k_{x_1} t_1$, $\beta_0\equiv k_{y_1}/k_{x_1}$, $\beta(T)\equiv\beta_0-RT$, $v_{x,y}\equiv\hat{u}_{x,y}/c_s$, $e\equiv -k_{x_1}\hat{S}'/(\partial_y S_0)$, $f\equiv\hat{P}'/P_0$, $\alpha\equiv P_0/\rho_0 c_s^2$, and $\xi\equiv g_0/k_{x_1}c_s^2$. We also note that the dimensionless measure of the characteristic frequency of pure internal gravitational waves, $\omega_0^2\equiv-(g_0/\rho_0)(\partial\rho_0/\partial S_0)_{P_0}(\partial_y S_0)$, can be readily defined as $W^2\equiv(\omega_0/c_s k_{x_1})^2$.

In this notation, the set of equations reduces to

$$\alpha\partial_T f=-i[v_x+\beta(T)v_y]+\xi v_y, \tag{8}$$

$$\partial_T e=v_y, \tag{9}$$

$$\partial_T v_x=-Rv_y-i\alpha f, \tag{10}$$

$$\partial_T v_y=-i\alpha\beta(T)f-W^2e+(1-\alpha)\xi f. \tag{11}$$

When gravity is absent ($W^2=\xi=0$) these equations [without Eq. (9)] reduce to the system describing plain sound waves in free shear flows [9]. Note that the IGW can be retained in the system by assuming a nonzero $W^2$. Furthermore, the coupling between IGW and SW will be nonzero even if the gravity-induced coupling ($\xi$) is small and negligible. Thus, without any fear of losing basic physics, we go ahead and neglect $\xi$ everywhere, and find the simplified system of equations [$F=i\alpha f$],

$$\partial_T F=v_x+\beta(T)v_y, \tag{12}$$

$$\partial_T e=v_y, \tag{13}$$

$$\partial_T v_x=-Rv_y-F, \tag{14}$$

$$\partial_T v_y=-\beta(T)F-W^2e. \tag{15}$$

The spectral energy density of the SFH may be defined as $E\equiv(v_x^2+v_y^2)/2+F^2/2+W^2e^2/2$, where the three terms correspond, respectively, to the fluid kinetic energy, the acoustic potential energy, and the internal-wave potential energy. The spectral energy density $E(T)$ satisfies the differential equation $\partial_T E=-Rv_x v_y$. When $R=0$ (the fluid at rest), $E(T)$ is conserved as expected.

In terms of a new variable $\psi(T)\equiv F-\beta(T)e[\partial_T\psi=v_x+Re]$, it is easy to transform Eqs. (12)–(15) into the following pair of second order differential equations:

$$\partial_{TT}\psi+\psi+\beta(T)e=0, \tag{16}$$

$$\partial_{TT}e+[W^2+\beta^2(T)]e+\beta(T)\psi=0, \tag{17}$$

representing two oscillators coupled through $\beta(T)$ [13], with $\omega_1\equiv1$ and $\omega_2(T)\equiv\sqrt{W^2+\beta^2(T)}$ as their respective eigenfrequencies. The presence of shear in the flow ($R\neq0$) ensures temporal variability of one of the uncoupled eigenfrequencies [$\omega_2(T)$] and of the *coupling coefficient* $\beta(T)$. Note that the time dependence of these quantities may be considered *adiabatic* when $R\ll1$ [9,10].

Fundamental vibrational frequencies of the coupled oscillators Eqs. (16) and (17) are equal to [13]

$$\Omega_\pm^2=\tfrac{1}{2}[\omega_1^2+\omega_2^2\pm\sqrt{(\omega_1^2-\omega_2^2)^2+4\beta^2}]. \tag{18}$$

Note that in the absence of gravity ($g_0=W=0$) $\Omega_+(T)\approx\sqrt{1+\beta^2(T)}$ reduces to the plain sound mode [9], while $\Omega_-(T)=0$, as it, certainly, should be.

Since the oscillation system, described by Eqs. (16) and (17), has two degrees of freedom its behavior may be determined by two functions $\psi(T)$ and $e(T)$. Note that all other physical quantities may be explicitly expressed in terms of $\psi$, $e$, and their first derivatives: $F=\psi+\beta(T)e$, $v_x=\partial_T\psi-Re$, and $v_y=\partial_T e$.

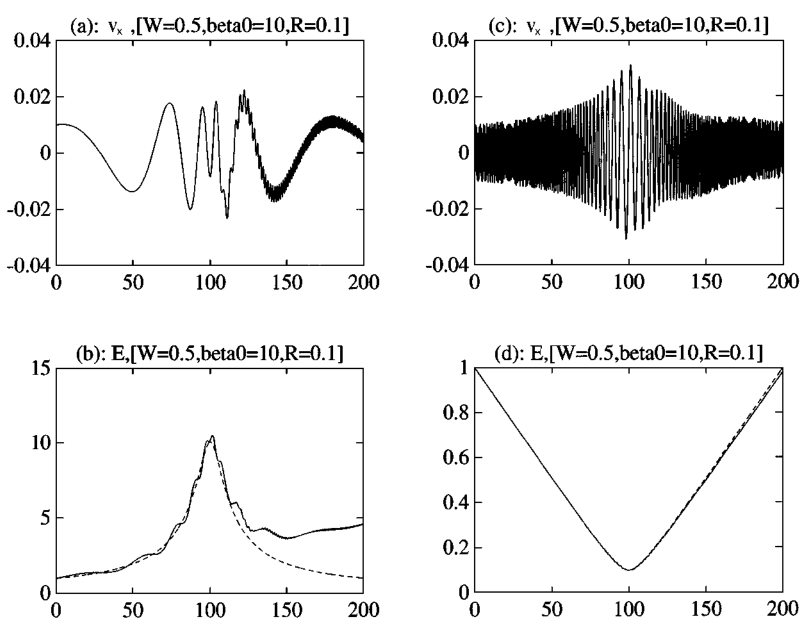


FIG. 1. The temporal evolution of the velocity $v_x(T)$ and energy $E(T)/E(0)$, respectively, for an initially pure IGW [(a) and (b)] and SW [(c) and (d)] modes. Dashed lines in (b) and (d) represent the $\Omega_-(T)/\Omega_-(0)$ (IGW) and $\Omega_+(T)/\Omega_+(0)$ (SW) curves, respectively, for initially excited modes. $\beta_0=10$, $R=0.1$, and $W=0.5$.

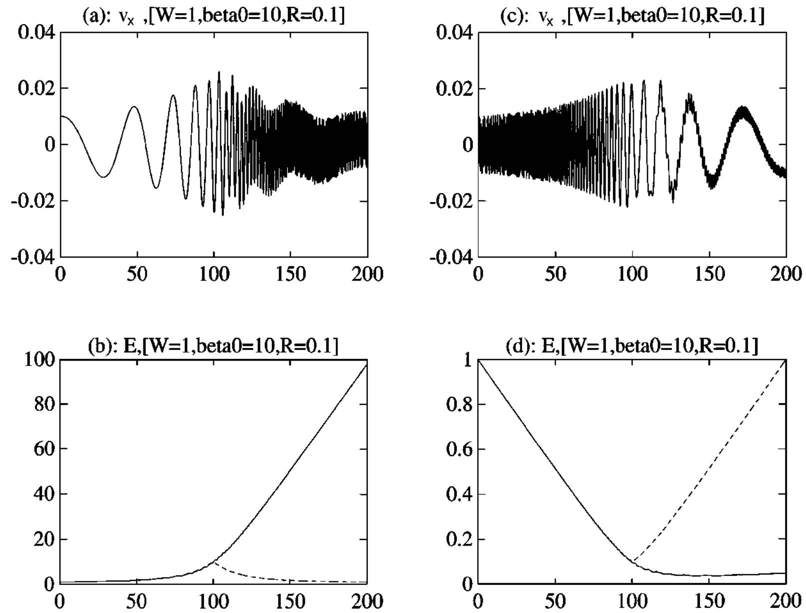


FIG. 2. The temporal evolution of the velocity $v_x(T)$ and energy $E(T)$, respectively, for an initially pure IGW [(a) and (b)] and SW [(c) and (d)] modes. Dashed lines in (b) and (d) represent the $\Omega_-(T)/\Omega_-(0)$ (IGW) and $\Omega_+(T)/\Omega_+(0)$ (SW) curves, respectively, for initially excited modes. $\beta_0=10$, $R=0.1$, and $W=1$.

The necessary conditions [14,10] for an effective energy exchange between two weakly coupled oscillators are the existence of a so-called ''degeneracy region,'' (DR) where $|\omega_1^2-\omega_2^2|\leqslant|\beta(T)|$, and that the DR should be ''passed'' slowly — the traversal time should be much greater than the period of the beats $|\partial_T\omega_2(T)|\ll|\beta(T)|$. The degeneracy region is in the neighborhood of $T_*\equiv\beta_0/R$, and $W=1$ leads to the most efficient mode coupling, and hence to the possibility of mutual transformation of the modes. It is straightforward to see that for the current problem, the existence of DR is ensured if $|\beta(T)|<1$. As regards the condition for $|\partial_T\omega_2(T)|$, in our case it reduces to the inequality $R\ll\sqrt{W^2+\beta^2(T)}$, which is true for all $T$ if $R\ll W$. Since $R\ll 1$, it is clear that for $W=1$, the conditon is always satisfied.

Regarding the ''adiabatic behavior'' of the modes, we should expect that the modes should normally follow the dispersion curves of their own: spectral energy density of either IGW $[E_-(T)]$ or SW $[E_+(T)]$ should be proportional to its corresponding frequency: $E_\pm\sim\Omega_\pm$ [9]. This mode of energy evolution, however, will not pertain in DR, where efficient transformation of one wave into the other occurs for $W=1$. For instance, the energy of an initially excited IGW mode increases approximately by the $E_-(T)\sim\Omega_-(T)$ law up to the vicinity of the point $T_*$, where it is partially transformed into SW. Afterwards, its energy evolution would still proceed adiabatically, but now according to the law $E_+(T)\sim\Omega_+(T)$.

One more, quite impressive, evolution regime can be realized when $R\ll\beta_0\ll 1$. In this particular case (with $W=1$), normal frequencies of $\psi$ and $e$ ''oscillators'' $[\Omega_+(T)$ and $\Omega_-(T)]$ are almost equal to each other and the coupling is inherently efficient. In this case beat modes will result: if initially, only one, say, the ''$e$ oscillator'' (i.e.,

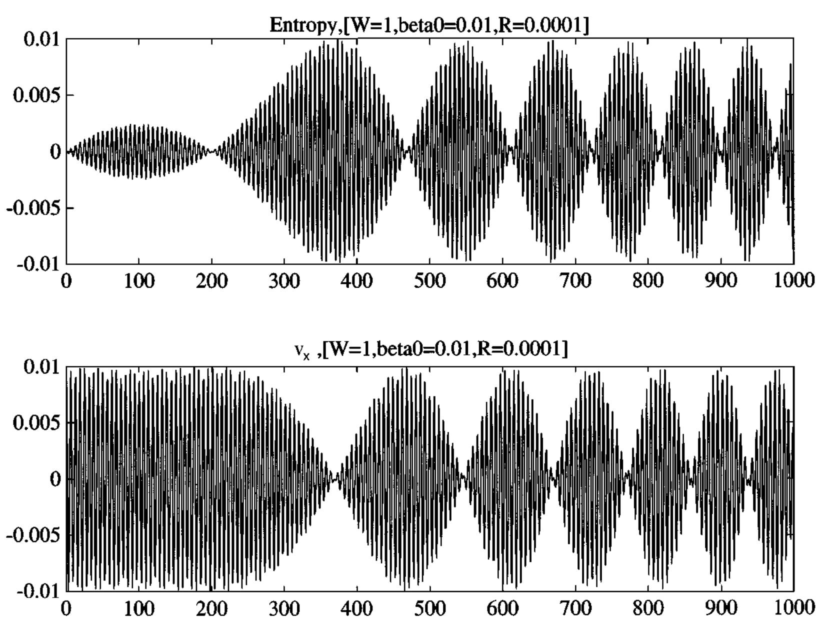


FIG. 3. Beat waves, displayed for $e(T)$ and $v_x(T)$ when $\beta_0=10^{-2}$, $R=10^{-4}$, and $W=1$.

$v_{x_0}=v_{y_0}=F_0=0$, and $e_0\neq 0$) is excited, beat waves with frequency $\Omega_b\equiv\Omega_+(T)-\Omega_-(T)$ will appear in time. Notice that the frequency is variable, and gets smaller and smaller after $T$ exceeds $T_*$.

In order to demonstrate the mutual transformation of IGW and SW with corresponding energy transfer between the modes, it is essential to choose initial conditions in such a way that at $T=0$, only one of the two modes is nonzero. Originally, we must calculate for $T=0$ the auxiliary quantities [13]: $\sigma_\pm=(\Omega_\pm^2-\omega_1^2)/\beta_0$.

For exciting pure IGW ($\Omega_-$ mode), we should choose $e_0=\sigma_-\psi_0$, and $\partial_T e_0=\sigma_-\partial_T\psi_0$. Recalling that $\psi_0=F_0-\beta_0 e_0$, $\partial_T\psi_0=v_{x_0}+Re_0$, and $\partial_T e_0=v_{y_0}$, we can simply take $F_0=e_0=0$ and an arbitrary $v_{x_0}$, and $v_{y0}=\sigma_- v_{x0}$. In exactly the same fashion we can excite pure SW with $F_0=e_0=0$, and $v_{y0}=\sigma_+ v_{x0}$.

The results of numerical calculations are partly presented in Figs. 1–3. They are in almost complete agreement with qualitative expectations.

In Figs. 1(a) [1(c)] and 11(b) [1(d)], we display the temporal evolution of the velocity $v_x(T)$ and energy $E(T)/E(0)$, respectively, for a pure IGW [SW] initial condition, and with the initial data $\beta_0=10$, $R=0.1$, and $W=0.5$. It is clearly seen that IGW [SW] evolves in the usual manner [following adiabatically the corresponding $\Omega(T)/\Omega(0)$ curve, presented by the dashed line] until it reaches DR ($T_*=100$ here), where a small portion of the other wave appears.

Figure 2 is a repetition of Fig. 1 with the notable difference that the resonant value of $W=1$ is taken. The mutual transformation of modes is now especially effective. The graphs show that there occurs almost complete transformation of IGW into SW and *vice versa*.

Finally, we display in Fig. 3 the results of numerical calculations for $e(T)$ and $v_x(T)$ for $\beta_0=10^{-2}$, $R=10^{-4}$, and $W=1$ chosen to favor beat wave generation. The graphs unambiguously show pronounced beat waves with a continuous back and forth energy transfer between the physical variables.

By studying a highly simple 2D model of a stratified fluid, we have explored the consequences of the shear-induced coupling between the internal gravity and the sound wave that leads to the mutual modal transformation, and to a corresponding energy transfer. Apart from the concrete novelty of the results obtained in this paper, the *main message* of this work is that the velocity shear may act like an effective ‘‘mixer’’ of the different waves sustainable in shear flows of arbitrary origin and constitution.

The authors wish to thank George Chagelishvili and Vazha Berezhiani for valuable discussions. A.D.R.'s research was supported, in part, by International Science Foundation (ISF) long-term research grant RVO 300. ADR's visit to ICTP was supported, in part, by the Committee for Science and Technology of Republic of Georgia.

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